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REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

The numbers in brackets are assigned according to the American Mathematical Society classification scheme. The 1991 Mathematics Subject Classification can be found in the annual subject index of *Mathematical Reviews* starting with the December 1990 issue.

1[65N30, 65N15]—*Superconvergence in Galerkin finite element methods*, by Lars B. Wahlbin, Lecture Notes in Mathematics, Vol. 1605, Springer, Berlin, 1995, xii+166 pp., $23\frac{1}{2}$ cm, softcover, \$33.00

These are the notes from a graduate seminar on superconvergence of finite element methods for second order scalar elliptic boundary value problems. Assuming a smooth exact solution, standard finite element methods for such problems provide optimal order approximation in the maximum norm for both the solution and its first derivatives. Thus, if finite elements based on piecewise polynomials of degree $d \geq 2$ are used, then the L^∞ norm of the error will converge to zero as $O(h^{d+1})$ as the meshsize h tends to zero, and the L^∞ norm of the gradient of the error will converge as $O(h^d)$. Simple approximation theory informs us that no better order of convergence can be achieved by any piecewise polynomial of degree d.

However, there is no obstruction to the error achieving higher order at isolated points. Indeed, under very general circumstances finite element errors can be shown to be of higher order when measured in Sobolev space norms of negative index. This indicates that the error is oscillatory, and thus must be smaller at some points than others. A point where the error achieves higher order than is possible globally is called a superconvergent point. For example, for a one-dimensional problem and continuous piecewise quadratic elements, the finite element solution is superconvergent at the mesh points and element midpoints, and its derivative is superconvergent at the two Gauss points in each element. Moreover, there can be no more superconvergent points for either the solution or its derivative, since an assumption to the contrary brings us into contradiction with approximation theory once again. These notes begin with an extensive study of such superconvergence in one dimension, including both the well-developed theory for merely continuous piecewise polynomials and more recent results (many due to the author) for smoother splines, in which case uniformity or symmetry restrictions are generally required on the mesh.

For higher-dimensional finite elements, superconvergence at readily identifiable points is a much more special phenomena and, except in the case of tensor product meshes and elements, the one-dimensional theory is of little guidance. However, some results are long known in the case of meshes with a high degree of uniformity. For example, on a uniform triangular mesh generated by three families of parallel lines, the element vertices and edge midpoints are superconvergent points when continuous piecewise quadratic elements are used. A major theme of these notes, and of the author's recent research in the area, is to relate such superconvergence behavior to the invariance of the mesh under the reflection through the superconvergent point. This approach leads to results for much more general meshes, though still requiring a high degree of symmetry. A second major theme is to localize the symmetry requirements to a neighborhood of the superconvergent points, with general meshes allowed far enough away from the points. The case of superconvergent behaviour of the L^2 projection is studied as well, since the techniques are similar to those used to study the finite element solution but analysis for the L^2 projection is somewhat simpler.

Other topics in the book include superconvergence by "trivial, or not so trivial" postprocesses, for example, the use of difference quotients on translation invariant meshes as superconvergent approximations of derivatives and the use of various averaging operators to achieve superconvergence; extensions to nonlinear problems; extensions to meshes which are smooth mappings of meshes with sufficient invariance properties; superconvergence results for boundary integral operators on one-dimensional boundary curves; and numerical studies to locate superconvergent points.

Superconvergence is presented in these notes as a fascinating and challenging area of mathematical analysis—the question of its significance in practical computation is not a major concern. The treatment is extensive, the references nearly exhaustive, the proofs complete, and the exposition clear and precise. The book will surely be appreciated by researchers wanting some guidance through the vast literature on superconvergence, and by readers who appreciate the intricate and subtle craft of technical finite element numerical analysis.

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2[35R30]—*Inverse problems in diffusion processes*, Heinz W. Engl and William Rundell (Editors), SIAM Proceedings Series, SIAM, Philadelphia, PA and GAMM, Regensburg, Germany, 1995, xii+232 pp., $25\frac{1}{2}$ cm, softcover, $58.00

The eleven papers in this book are based on some of the invited talks at a conference held in Austria during the summer of 1994.

Three papers deal with the inverse heat conduction problem: one by Beck on the function specification method, one by Eldén on a numerical method using Tikhonov regularization, and one by Murio, Liu, and Zheng on a mollification method for stabilizing the inverse problem.

Two papers deal with theoretical aspects of regularization: Seidman with general considerations in dealing with ill-posed problems, and Chavent with recent results on the regularization of nonlinear least squares problems.

Three papers deal with the determination of unknown coefficients in second-order parabolic equations. In particular, Isakov considers identifiability from lateral and final data; Lowe and Rundell consider identifiability using boundary fluxes from interior sources; and it first reviews the literature and then discusses methods based on transforming the inverse problem to one involving a nonlocal functional.

The paper by Kunisch is a survey of some recent work on numerical methods for estimating the coefficients of elliptic equations.

The paper by Vainikko uses projection discretization schemes with Tikhonov regularization to deal with an inverse problem in groundwater filtration.